SPACES OF ELLIPTIC DIFFERENTIALS

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ABSTRACT. We study modular fibers of elliptic differentials, i.e. investigate spaces of coverings $(Y,\tau) \to (\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}i, dz)$. For genus 2 torus covers with fixed degree we show, that the modular fibers $\mathscr{F}_d(1,1)$ are connected torus covers with Veech group $\mathrm{SL}_2(\mathbb{Z})$. Using results of Eskin, Masur and Schmoll [EMS] we calculate $\chi(\mathscr{F}_d(1,1))$ and the parity of the spin structure of the quadratic differential $(\mathscr{F}_d(1,1)/(-\mathrm{id}), q_d)$. We state and apply formulæ for the asymptotic quadratic growth rates of various types of geodesic segments on $(Y,\tau) \in \mathscr{F}_d(1,1)$. The quadratic growth rates are expressed in terms of the $\mathrm{SL}_2(\mathbb{Z})$ orbit closure of (Y,τ) in $\mathscr{F}_d(1,1)$ and the flat geometry of $\mathscr{F}_d(1,1)$.
These are extended notes from a talk the author gave during the *Activity on Algebraic and Topological Dynamics* at the Max-Planck-Institute for Mathematics, Bonn summer 2004.

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1. INTRODUCTION

Motivation and Background. If we want to find the length distribution of isotopy classes of closed geodesics on the flat torus $\mathbb{T}^2 := \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}i \cong \mathbb{R}^2/\mathbb{Z}^2$, the answer is easily obtained by counting integer lattice points in the plane:

$$N(\mathbb{T}^2, T) := |\{(x,y) \in \mathbb{Z}^2 : \ \gcd(x,y) = 1, \sqrt{x^2+y^2} < T\}| \sim \frac{\pi}{\zeta(2)} T^2 = \frac{6}{\pi} T^2.$$

The factor $\frac{1}{\zeta(2)}$ arises, if one counts *primitive* geodesics (see [EM98]) including their direction, counting of geodesics ignoring direction requires the weight $\frac{1}{2\zeta(2)}$. Primitive geodesics (with direction) are represented by integer lattice points in $\mathbb{R}^2$ which are *visible from the origin*. We like to ask the same question for a (branched) covering $\pi : X \to \mathbb{C}/\Lambda$, $\Lambda \subset \mathbb{C}$ a lattice. The necessary (complex) geometric structure on X is a *holomorphic differential* ω obtained by pulling back the differential dz on

Date: February 2, 2008.
2000 MATHEMATICS SUBJECT CLASSIFICATION: 14H15, 14H52, 30F30, 30F60, 37C35, 58D15, 58D27

$\mathbb{C}/\Lambda$. We call the pair $(X, \omega = \pi^* dz)$ *elliptic differential*. Using ω one identifies X locally with regions in the complex plane by coordinates of the shape

$$z_{p_0}(p) = \int_{p_0}^{p} \omega$$

away from the zero-set $Z(\omega)$ of ω. With respect to these charts coordinate changes are translations, in particular the Euclidean metric pulls back to (X, ω) and defines a global Euclidean metric on $X - Z(\omega)$.

We are mainly interested in the following geodesic segments on (X, ω):

- (isotopy classes of) closed geodesics $Cyl(\omega)$ and
- saddle connections $SC(\omega)$, these are geodesic segments starting and ending at zeros of ω, without hitting a zero in between.

We like to study the asymptotic quadratic growth rate of these geodesic segments on (X, ω) with respect to the Euclidean metric defined by ω, i.e. we look at the number

$$N_{cyl}(\omega, T) := \left| \left\{ \gamma \in Cyl(\omega) : \int_\gamma |\omega| < T \right\} \right|. \tag{1}$$

A fundamental result of Masur (for a new version see [EM98]) says

Theorem 1. [M3, M4] *For any translation surface (X, ω) there are constants, such that for $T >> 0$*

$$0 < c_1 T^2 < N_{cyl}(\omega, T) < c_2 T^2.$$

The same is true for the set of saddle connections $SC(\omega)$ on (X, ω) (eventually with different constants c_i).

Surprisingly in various cases [V2, EMS, EMZ, McM3, EMM], there is an asymptotic quadratic formula

$$N_{cyl}(\omega, T) \sim \frac{\pi}{\zeta(2)} c_{cyl}(\omega) T^2.$$

Moreover: in all cases known so far the constant can be computed [V3, EMS], or at least expressed in terms of geometrical data of the moduli space of Abelian differentials where (X, ω) belongs too [EMZ]. For general differentials (X, ω) it is not known that the various asymptotic constants $c_*(\omega)$ exist. In the case of elliptic differentials however, it is well known (see [EMS, EMM, S1]) that all asymptotic constants, including $c_{cyl}(\omega)$ and $c_{SC}(\omega)$ are well defined.

To introduce the main objects we give an alternative, more geometric description of translation surfaces.

Translation surfaces by gluing polygons. Take a finite set of polygons $P_1, ..., P_n$ in the complex plane $\mathbb{C}$ with boundary components ∂P_i oriented counter-clockwise and for each edge $\mathfrak{a} \in \cup_i \partial P_i$ there is a unique translation $\mathfrak{t}_\mathfrak{a} \neq 0$ such that $\mathfrak{a} + \mathfrak{t}_\mathfrak{a} = -\mathfrak{b} \in \cup_i \partial P_i$. Identifying pairs of edges $\mathfrak{a}$ and $\mathfrak{a} + \mathfrak{t}_\mathfrak{a}$ gives a compact surface X which is by construction a *translation surface* with flat metric induced by the Euclidean metric on $\mathbb{C}$. Moreover a line field in direction $\theta \in S^1$ on $\mathbb{C}$ induces a foliation $\mathscr{F}_\theta(X)$ on X. Finally the differential dz descends to X (vertices of the polygons removed) and defines a holomorphic differential ω on X with zeros located in the vertices of the P_i. Really important is the following group operation:

$\mathbf{SL_2}(\mathbb{R})$ action on translation surfaces. Take the linear operation of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^2 \cong \mathbb{C}$ and choose $A \in \mathrm{SL}_2(\mathbb{R})$. Then the set $\cup_i AP_i \subset \mathbb{C}$ with the identification $\cup_i \partial AP_i \ni A \cdot \mathfrak{a} \leftrightarrow A \cdot \mathfrak{a} + A \cdot \mathfrak{t}_{\mathfrak{a}} = -A \cdot \mathfrak{b} \in \cup_i \partial AP_i$ is a translation surface $A \cdot X$, a deformation of X. In this way we obtain a $\mathrm{SL}_2(\mathbb{R})$-action on the set of translation surfaces.

Equivalence relation. Two translation surfaces X, Y are *equivalent*, if there exists a translation diffeomorphism $\phi : X \to Y$, i.e. $\mathrm{D}\,\phi = \mathrm{id}$ in polygonal coordinates above. The *moduli space* of equivalence classes of translation surfaces can be identified with the moduli space $\Omega_1\mathscr{M}_g$ of genus g Abelian differentials with normalized area (given by equation 2).

Elliptic differentials in genus 1 – Lattice surfaces. Take an elliptic differential (X, ω) of genus 1, i.e. a Riemann surface X of genus one with a holomorphic one form ω. For simplicity we assume (X, ω) has normalized area:

$$\mathrm{area}_\omega(X) = \frac{i}{2}\int_X \omega \wedge \bar{\omega} = 1. \tag{2}$$

The *absolute periods*

$$\mathrm{Per}(\omega) := \left\{ \int_\gamma \omega : \gamma \in H_1(X;\mathbb{Z}) \right\} \subset \mathbb{R}^2 \cong \mathbb{C}$$

of (X, ω) define a lattice in $\mathbb{R}^2$. In particular the elliptic differential $(\mathbb{C}/\,\mathrm{Per}(\omega), dz)$ has the same absolute period lattice as (X, ω) and in natural charts

$$z(p) = \int_{p_0}^{p} \omega$$

we see that locally $dz = \omega$. This in turn implies, up to orientation

$$(X, \omega) \cong (\mathbb{C}/\,\mathrm{Per}(\omega), \pm dz) \leftrightarrow \mathrm{Per}(\omega),$$

i.e. up to sign each elliptic differential can be identified with a lattice $\Lambda \in \mathbb{C}$.

Now represent the flat torus $\mathbb{R}^2/\mathbb{Z}^2$ by the square Q with vertices $(0,0), (1,0), (1,1), (0,1) \in \mathbb{Z}^2$ and take its image $A \cdot Q \subset \mathbb{R}^2$ under $A \in \mathrm{SL}_2(\mathbb{R})$. Identifying parallel sides of the parallelogram $A \cdot Q$ defines a new torus $\mathbb{T}^2_A$. Moreover the edges of $A \cdot Q$ define a lattice $\Lambda_A = A \cdot \mathbb{Z}^2$, such that

$$A \cdot \mathbb{T}^2 = \mathbb{T}^2_A = \mathbb{R}^2/\Lambda_A.$$

It is clear that $\Lambda_A = A \cdot \mathbb{Z}^2 = \mathbb{Z}^2$ if and only if $A \in \mathrm{SL}_2(\mathbb{Z})$. To discover a translation homeomorphism

$$\mathbb{T}^2_A \to \mathbb{T}^2 \quad \text{for } A \in \mathrm{SL}_2(\mathbb{Z})$$

think of $\mathbb{R}^2$ being square-tiled by copies of Q. Now consider $A \cdot Q \subset \mathbb{R}^2$ and cut $\mathbb{R}^2$ along the edges of the square-tiling by Q. Then reassemble Q by translating the pieces obtained from cutting $A \cdot Q$. Note that the $A \in \mathrm{SL}_2(\mathbb{R})$ for which vertices of $A \cdot Q$ become vertices of Q, are exactly the $A \in \mathrm{SL}_2(\mathbb{Z})$. Thus the moduli space $\mathscr{E}_1$ of (normalized) elliptic differentials or translation tori (with a direction) equals

$$\mathscr{E}_1 = \mathrm{SL}_2(\mathbb{R}) \cdot \mathbb{T}^2 \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z}).$$

We see that the *stabilizer* $\{A \in \mathrm{SL}_2(\mathbb{R}) : A \cdot \mathbb{T}^2 \cong \mathbb{T}^2\} \cong \mathrm{SL}_2(\mathbb{Z})$ is a *lattice* in $\mathrm{SL}_2(\mathbb{R})$. For general (X, ω) one writes

$$\mathrm{SL}(X, \omega) := \{A \in \mathrm{SL}_2(\mathbb{R}) : A \cdot (X, \omega) \cong (X, \omega)\}$$

and calls this stabilizer the *Veech group* of (X,ω). It is already remarkable that there are translation surfaces (X,ω) with a lattice stabilizer $\mathrm{SL}(X,\omega) \subset \mathrm{SL}_2(\mathbb{R})$ which are not elliptic differentials. Translation surfaces with lattice stabilizer are called *lattice surfaces* or *Veech surfaces* in honor of W. Veech who found the first series of lattice surfaces, which are not elliptic differentials, by unfolding billiards in the regular n-gon [V2, V3]. Recently C. T. McMullen [McM1] and Kariane Calta [C] discovered that L-shaped polygons with a certain algebraic condition on the length of their sides are also lattice surfaces. Infinitely many Veech surfaces constructed from L-shaped tables, or from regular n-gons, are *not* elliptic differentials.

Action of $\mathbf{SL}(\mathbf{X},\,\omega)$ on coverings. Given a lattice surface (X,ω) and a branched covering $\pi : (Y,\tau = \pi^*\omega) \to (X,\omega)$, a deformation of (Y,τ) by $A \in \mathrm{SL}(X,\omega)$ is again a cover of (X,ω). Thus we get an operation of $\mathrm{SL}(X,\omega)$ on coverings of (X,ω)

The set of all coverings of $\mathbb{T}^2$ of *fixed genus, fixed degree and branch points of fixed order* has a natural manifold structure. Moreover 2-dimensional components of this *modular fiber* are elliptic differentials and cover $(\mathbb{T}^2, dz)$. We denote 2-dimensional modular fibers by $(\mathscr{F}, \omega_{\mathscr{F}})$ or short $\mathscr{F}$.

The goal of this paper is to describe a method relating asymptotic constants of an elliptic differential (X,ω) to the translation geometry and topology of the modular fiber $\mathscr{F}$ containing (X,ω). This approach to elliptic differentials can be easily extended to covers of lattice- or Veech-surfaces. We also give examples of modular fibers $\mathscr{F}$ and an easy calculation of asymptotic constants using our formula. The asymptotic constants of $(X,\omega) \in \mathscr{F}$ depend on the *translation geometry* of $(\mathscr{F}, \omega_{\mathscr{F}})$ and on the orbit closure $\overline{\mathrm{SL}_2(\mathbb{Z}) \cdot (X,\omega)} \subset \mathscr{F}$. Since it is sometimes hard to characterize a connected component of the modular fiber by topological and geometrical invariants we will follow another way.

The modular fiber. Assume (X,ω) is a lattice surface with lattice group $\mathrm{SL}(X,\omega)$ We define a space of coverings $\mathscr{F}_{\omega,\tau}$ using a given cover $(Y,\tau) \to (X,\omega)$ and the $\mathrm{SL}(X,\omega)$ action on covers of (X,ω) as

$$\mathscr{F}_{\omega,\tau} := \overline{\mathrm{SL}(X,\omega) \cdot (Y,\tau)}.$$

There are two possible ways of taking a closure of $\mathrm{SL}(X,\omega) \cdot (Y,\tau)$:

1. inside the space of differentials with fixed number and orders of zeros, or
2. including all limiting surfaces, thus *degenerated* surfaces appear in $\mathscr{F}_{\omega,\tau}$.

For this paper we will assume $\mathscr{F}_{\omega,\tau}$ is obtained with respect to the *first closure.* In our particular cases it is not hard to see that $\mathscr{F}_{\omega,\tau}$ is always an open complex space. If $(Y,\tau) \to (X,\omega)$ has n-branch points one can show [S2], that there is a natural map

$$\pi_* : \mathscr{F}_{\omega,\tau} \to X^n.$$

This map is either a covering of X^n, or a covering of an $\mathrm{SL}(X,\omega)$-invariant subspace of X^n.

To avoid technical difficulties, we only discuss covers of $\mathbb{T}^2$ branched over exactly 2 named points. Tracking the two branch points on the base torus while deforming a cover in the modular fiber gives a map to $\mathbb{T}^2 \times \mathbb{T}^2 - \{[x,x] : [x] \in \mathbb{T}^2\}$. Since $\mathbb{R}^2/\mathbb{Z}^2$ acts by translations on $\mathbb{T}^2$ and on the modular fiber we divide out this torus action.

Equivalently we might assume one of the branch points is fixed, say at $[0] \in \mathbb{T}^2$ and obtain a covering map $\mathscr{F}_\tau \to \mathbb{T}^2 - \{[0]\}$. Here we have simplified $\mathscr{F}_\tau := \mathscr{F}_{\tau,\omega}$, because $\omega = dz$. Now the translation structure of $\mathbb{T}^2$ pulls back to $\mathscr{F}_\tau$ and we obtain an elliptic differential

$$(\mathscr{F}_\tau, \omega_\tau) := (\mathscr{F}_\tau, \pi^* dz)$$

which by $\mathrm{SL}_2(\mathbb{Z})$ invariance is a union of lattice surfaces with Veech group $\mathrm{SL}_2(\mathbb{Z})$. Now we can take the *unique compactification* $\mathscr{F}_\tau^c$ of $\mathscr{F}_\tau$ which makes the continuation of ω_τ to $\mathscr{F}_\tau^c$ *holomorphic*.

Degenerated translation surfaces. To understand the geometry of $\mathscr{F}_{\omega,\tau}$ one needs to look at degenerated surfaces X_{deg}. These are just deformed Abelian differentials obtained by moving two or more cone points into one point. There are cases when the degeneration process leads to a union of two or more translation surfaces, which are connected in some special points only. In algebraic geometry degenerated surfaces are known as *stable, nodal curves*.

Example: The space $\mathscr{F}_d(1,1)$ of elliptic differentials (X,ω) with *two distinguished zeros* $z_1 \neq z_2$ of *order* 1 each, $\mathrm{Per}(\omega) = \mathbb{Z} \oplus \mathbb{Z}i$ and $\deg(\pi) = d$, $\pi : (X,\omega) \to \mathbb{C}/\operatorname{Per}(\omega)$ covers $\mathbb{T}^2 - \{[0]\}$ and carries a natural Abelian differential ω_d. If we take a differential in $\mathscr{F}_d(1,1)$ and collapse its two cone points into one, we obtain a surface with one cone point of order 3, or a degenerated differential. Differentials with one order 3 cone point in turn are order 3 cone points of $(\mathscr{F}_d(1,1), \omega_d)$. In [EMS] we show that ω_d has exactly

$$|Z(\omega_d)| = \frac{3}{8}(d-2)d^2 \prod_{p|d}(1 - \frac{1}{p^2}) \tag{3}$$

zeros, all of order 2. There are other surfaces in the compactification of $\mathscr{F}_d(1,1)$, which are degenerated in the sense above: these surfaces are either two tori identified in one point, or a torus with with two points identified. The total number of degenerated surfaces X_{deg} in $\mathscr{F}_d(1,1)$ is

$$N_{deg}(d) = \frac{1}{24}(5d+6)d^2 \prod_{p|d}(1 - \frac{1}{p^2}) \quad \text{for } d \geq 3 \tag{4}$$

and $N_{deg}(2) = 4$. This is the order of a union of $\mathrm{SL}_2(\mathbb{Z})$-orbits on $\mathscr{F}_d(1,1)$. For the rest of the paper we use the *Euler φ function* and the *Dedekind ψ function*:

$$\varphi(d) := d \prod_{p|d}\left(1 - \frac{1}{p}\right), \quad \psi(d) := d \prod_{p|d}\left(1 + \frac{1}{p}\right) \tag{5}$$

to write

$$d^2 \prod_{p|d}(1 - \frac{1}{p^2}) = \varphi(d)\psi(d).$$

Remark. The counting formulæ 3 and 4 were independently discovered by Kani [Ka1, Ka2] with motivation and tools from algebraic geometry.

2. Results and applications

With the conventions of the previous example, we establish

Theorem 2. *The modular fiber* $(\mathscr{F}_d(1,1),\omega_d)$ *is connected. In particular the Veech group of* $(\mathscr{F}_d(1,1),\omega_d)$ *is*

$$\mathrm{SL}(\mathscr{F}_d(1,1),\omega_d)\cong \mathrm{SL}_2(\mathbb{Z}).$$

Remark. Connectedness of $\mathscr{F}_d(1,1)$ was already established by W. Fulton [Fu], as the author learned from C. T. McMullen. However at the end of the paper we prove connectedness of $\mathscr{F}_d(1,1)$ by using that it is a $\mathrm{SL}_2(\mathbb{Z})$-orbit of a torus-cover in moduli space.

All modular fibers $\mathscr{F}_d(1,1)$ admit an involution σ with linear part $-\mathrm{id}\in\mathrm{SL}_2(\mathbb{Z})$, thus we can consider the the double cover

$$\mathrm{pr}_\sigma:\mathscr{F}_d(1,1)\to\mathscr{F}_d(1,1)/\sigma$$

The quotient *quadratic differential* $(\mathscr{F}_d(1,1)/\sigma,q_d)$, or short $\mathscr{F}_d(1,1)/\sigma$, parameterizes (normalized) degree d elliptic differentials (Y,τ) with two *un*-distinguishable cone-points of order 1. Now any $(Y,\tau)\in\mathscr{F}_d(1,1)/\sigma$ admits a hyperelliptic involution, which interchanges its two cone points. In particular: distinguishing cone points destroys the hyperelliptic involution of (Y,τ).

Corollary 1. *We have:*

$$\begin{aligned}\chi(\mathscr{F}^c_d(1,1)) &= -\frac{3}{4}(d-2)\varphi(d)\psi(d)\quad \textit{for}\ \ d\geq 2\ \ \textit{and}\\ \chi(\mathscr{F}^c_d(1,1)/\sigma) &= -\frac{1}{12}(d-6)\varphi(d)\psi(d)\quad \textit{for}\ \ d\geq 3,\ \textit{while}\end{aligned}\tag{6}$$

$\chi(\mathscr{F}^c_2(1,1)/\sigma)=2$. *In particular* $\mathscr{F}^c_d(1,1)$ *is hyperelliptic, if and only if* $d=2,3,4$ *and* $d=5$. *The surface* $\mathscr{F}^c_d(1,1)/\sigma$ *is a torus if and only if* $d=6$.

Moreover the parity Ψ *of the spin structure defined by the meromorphic quadratic differential* q_d *on* $\mathscr{F}_d(1,1)/\sigma$ *is*

$$\Psi(q_d)=\frac{|\chi(\mathscr{F}^c_d(1,1)/\sigma)|}{2}\mod 2\equiv \tag{7}$$

$$\equiv\begin{cases}1 & \textit{if}\ \ d=2,3,4,5\\ 0 & \textit{if}\ \ d=2n\geq 6\\ \frac{1}{24}\varphi(d)\psi(d)\mod 2 & \textit{if}\ \ d=2n+1\geq 7.\end{cases}$$

Remark. The proof of these identities appears at the end of the paper. Let $\mathscr{H}:=\{z\in\mathbb{C}:\mathrm{Im}(z)>0\}$ be the *Poincaré upper half plane* and $\Gamma(d)$ the *principal congruence subgroup of level* d. In [Ka1, Ka2] E. Kani describes the Hurwitz-Scheme of genus 2 elliptic differentials and found (over $\mathbb{C}$) a description as an open subscheme of

$$X(d):=\Gamma(d)\backslash\mathscr{H}\quad !$$

Here the quotient space $X(d)$ is obtained by considering the action of $\mathrm{SL}_2(\mathbb{Z})$ by rational transformations on $\mathbb{H}$. The advantage of looking at the affine linear action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathscr{F}_d(1,1)/\sigma$ is, that it commutes with the $\mathrm{PSL}_2(\mathbb{Z})$-action on surfaces parameterized by $\mathscr{F}_d(1,1)/\sigma$.

The results above allow to describe asymptotic quadratic growth constants in terms

of the modular fiber. Quadratic growth rates are best expressed in terms of a *Siegel-Veech constant* [V4, EM98, EMZ]:

$$\frac{\pi}{\zeta(2)} \cdot c_{cyl}(\alpha) := \lim_{T\to\infty} \frac{N(Cyl(\alpha),T)}{T^2}$$

where

$$N(Cyl(\alpha),T) := |\{hol(c) \subset \mathbb{R}^2 : c \in Cyl(\alpha)\} \cap \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq T^2\}|$$

Here $\mathrm{hol}(c) = \int_\gamma \alpha$ and γ is any geodesic loop around the core of c.

Theorem 3 **(Cylinders).** [S2, S3] *Assume $(\mathscr{F}_\tau, \alpha_\tau)$ is a modular fiber parameterizing elliptic differentials (S,α) with exactly 2 cone points. Suppose the horizontal foliation of $\mathscr{F}_\tau$ decomposes into open cylinders $\mathscr{C}_1, ..., \mathscr{C}_{n_{\mathscr{F}}}$ of periodic regular leaves, bounded by singular leaves $\partial^{top}\mathscr{C}_1, ..., \partial^{top}\mathscr{C}_{n_{\mathscr{F}}}$. Then every elliptic differential $(S,\alpha) \in \mathscr{C}_i$ has a completely periodic horizontal foliation where the number of cylinders, say n_i, depends only on $\mathscr{C}_i$. The horizontal cylinders of (S,α) have width $w_{i,1}, ..., w_{i,n_i}$ independent of $(S,\alpha) \in \mathscr{C}_i$. If $(S,\alpha) \in \mathscr{F}_\tau$ has infinite $\mathrm{SL}_2(\mathbb{Z})$ orbit, its Siegel-Veech constant is:*

$$c_{cyl}(\alpha) = \frac{1}{\mathrm{area}(\mathscr{F}_\tau)} \sum_{i=1}^{n_{\mathscr{F}}} \sum_{k=1}^{n_i} \frac{\mathrm{area}(\mathscr{C}_i)}{w_{i,k}^2}. \tag{8}$$

If the $\mathrm{SL}_2(\mathbb{Z})$ orbit $\mathscr{O}_\alpha := \{A \cdot (S,\alpha) : \ A \in \mathrm{SL}_2(\mathbb{Z})\} \subset \mathscr{F}_\tau$ of (S,α) is finite, we have

$$c_{cyl}(\alpha) = \frac{1}{|\mathscr{O}_\alpha|} \sum_{i=1}^{n_{\mathscr{F}}} \left(\sum_{k=1}^{n_i} \frac{|\mathscr{O}_\alpha \cap \mathscr{C}_i|}{w_{i,k}^2} + \sum_{k=1}^{m_i} \frac{|\mathscr{O}_\alpha \cap \partial^{top}\mathscr{C}_i|}{w_{i,k}^2} \right). \tag{9}$$

Remark. *Connectedness of the modular fiber is not necessary to obtain this Theorem. Note however that $(\mathscr{F}_\tau, \alpha_\tau)$ is a union of surfaces if it is not connected. As for differentials contained in $\mathscr{C}_i$, the number and width of cylinders contained in the horizontal foliation of $(S,\alpha) \in \partial^{top}\mathscr{C}_i$ depends on $\partial^{top}\mathscr{C}_i$ only.*

Since asymptotic constants for cylinders on $(S,\alpha) \in \mathscr{F}_\tau$ depend on the cylinder-decomposition of $\mathscr{F}_h(\mathscr{F}_\tau)$, it is not surprising that the asymptotic constants for saddle connections connecting the two different cone points of (S,α) depend on the *saddle connections* of $\mathscr{F}_h(\mathscr{F}_\tau)$. Note, that we consider *degenerated surfaces* in the closure of $\mathscr{F}_\tau$ as *marked* points of $\mathscr{F}_\tau$ and therefore find typically more saddle connections in $\mathscr{F}_h(\mathscr{F}_\tau)$ as we expect from recognizing only cone points.

Denote the set of saddle connections contained in $\mathscr{F}_h(\mathscr{F}_\tau)$ by $SC_h(\mathscr{F}_\alpha)$ and note that *as a set of singular leaves*

$$SC_h(\mathscr{F}_\alpha) = \bigcup_{i=1}^{n} \partial^{top}\mathscr{C}_i.$$

Let us take $(S,\alpha) \in s \in SC_h(\mathscr{F}_\alpha)$ and deform (S,α) along s into the right $(+)$ or left $(-)$ endpoint of s in $\mathscr{F}_\alpha^c$. That means we degenerate (S,α) into a cone point or a point representing a degenerated surface of $\mathscr{F}_\alpha^c$. Tracking the family of deformed surfaces we see that along the deformation of (S,α) we degenerate m_s^+ (m_s^-) horizontal saddle connections of length s_α^+ (s_α^- respectively) on (S,α). Note that $s_\alpha^\pm$ equals the distance of $(S,\alpha) \in s$ to the right $(+)$ or left $(-)$ endpoint of s.

If o_1 and o_2 are the orders of the two zeros of α then the maximal number m of

saddle connection which can be killed by one deformation is $\min(o_1, o_2)$. To keep the following statement as elementary as possible, we name all the degenerated points and cone points of $(\mathscr{F}_\tau^c, \omega_\tau)$ and assume the list is given by $z_1, ..., z_{n_\tau}$. Associated to this list we get a list $o_1, ..., o_{n_\tau}$ of *orders* of the z_i and a list $m_1^+, ..., m_{n_\tau}^+$ of *multiplicities*, telling us how many saddle connections disappear while degenerating a surface into z_i from the right along a horizontal saddle connection. By walking along a small circle around the (cone-)point $z_i \in \mathscr{F}_\tau^c$ one can see that m_i^+ is well-defined, i.e. the same for each horizontal saddle connection s terminating in z_i.

Before we state the Theorem, we like to mention that there are asymptotic constants (see [S3]) which reflect finer properties of (S, α) and $\mathscr{F}_\tau$, for instance one can use different weights $m_s^\pm$ associated to topological/geometrical properties of the surfaces represented by the special points $z_i \in \mathscr{F}_\tau$ (see [S3]). One can restrict to certain subsets of the set of cone points or the set of horizontal saddle connections in $\mathscr{F}_\tau$ too.

Theorem 4 (**Saddle connections).** [S2, S3] *With the assumptions and notations of Theorem 3, we find for the asymptotic quadratic growth rate $c_\pm(\alpha)$ for saddle connections on $(S, \alpha) \in \mathscr{F}_\tau$ connecting the two different cone points of (S, α)*

$$c_\pm(\alpha) = \frac{2}{|\mathscr{O}_\alpha|} \sum_{s \in SC_h(\mathscr{F}_\alpha)} \sum_{(Z,\nu) \in \mathscr{O}_\alpha(s)} \frac{m_s^+}{(s_\alpha^+)^2}. \tag{10}$$

with $\mathscr{O}_\alpha(s) := \mathscr{O}_\alpha \cap s$ in the finite orbit case. For generic (S, α) we find

$$c_\pm(\alpha) = \frac{2\zeta(2)}{\mathrm{area}(\mathscr{F}_\alpha)} \sum_{i=1}^{n_\tau} m_i^+ \widehat{o}_i \quad \text{where } \widehat{o}_i = o_i + 1. \tag{11}$$

Remark. The straightforward generalization of the above Theorem [S3] includes: modular fibers of *higher dimension*, *arbitrary lattice group* $\mathrm{SL}(X, \omega)$ and *disconnected fibers* $\mathscr{F}_{\tau,\omega}$.

Depending on the specific problem, the formulæ presented here tie the evaluation of Siegel-Veech constants of an elliptic differential $(X, \omega) \in \mathscr{F}$ to

- the counting of certain types degenerated surfaces in the closure of the modular fiber $\mathscr{F}$
- the counting of cone points of $\omega_{\mathscr{F}}$
- the classification of finite $\mathrm{SL}_2(\mathbb{Z})$-orbits in $\mathscr{F}$.
  To determine the constants for saddle connections on lattice elliptic differentials one needs to know
- the intersection of a particular $\mathrm{SL}_2(\mathbb{Z})$ orbit with $SC_h(\mathscr{F}) = \bigcup_{i=1}^n \partial^{top}\mathscr{C}_i$.

A basic example. To apply the whole method we take the example of two marked tori, worked out by the author in [S1]. Take the torus $\mathbb{T}^2 = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}i \cong \mathbb{R}^2/\mathbb{Z}^2$ marked in two points. We assume one of the marked points is $[0] := 0 + \mathbb{Z}^2$, if the other is $[m] \neq [0]$ we write

$$\mathbb{T}^2_{[m]} = (\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}i, [0], [m]) \cong (\mathbb{R}^2/\mathbb{Z}^2, [0], [m])$$

The moduli space of 2-marked tori is simply the torus $\mathbb{T}^2 - \{[0]\}$ if we agree to distinguish the marked points.

Now the horizontal foliation of $\mathbb{T}^2 - \{[0]\}$ consists of one cylinder $\mathscr{C}$ (of height and

width one) and one saddle connection $\partial^{top}\mathscr{C}$ connecting $[0]$ with itself. Now the horizontal foliation of the torus $\mathbb{T}_{[m]}$ contains

- two cylinders of width one if $[m] \in \mathscr{C}$ and
- one cylinder of width one if $[m] \in \partial^{top}\mathscr{C}$

Formula 8 then implies that the asymptotic quadratic constant $c_{cyl}(gen)$ for isotopy classes of periodic trajectories for the generic two marked torus cover is 2 (which is of course easy to see without a fancy formula). Now a surface or point in $\mathscr{F}_1(0,0)$ is generic if and only if it has infinite $\mathrm{SL}_2(\mathbb{Z})$ orbit and these are exactly the irrational points in $\mathbb{T}^2 - \{[0]\}$, i.e. the set $\mathbb{T}^2 - \mathbb{Q}^2/\mathbb{Z}^2$.

Finite orbit case: torsion points on $\mathbb{T}^2$. The set of torsion points of $\mathbb{T}^2$ is the kernel of the multiplication homomorphism

$$\mathbb{T}^2[n] := \ker(\mathbb{T}^2 \xrightarrow{n} \mathbb{T}^2) = \frac{1}{n}\mathbb{Z}^2/\mathbb{Z}^2, \quad \text{where } n : [z] \mapsto [nz].$$

It is not hard to see that the $\mathrm{SL}_2(\mathbb{Z})$-orbit $\mathscr{O}_n$ of $\frac{1}{n} \in \mathbb{T}^2$ is

$$\mathscr{O}_n = \left\{ \left[\frac{a}{n} + i\frac{b}{n}\right] \in \mathbb{T}^2 : \ a, b, n \in \mathbb{Z} \text{ with } \gcd(a,b,n) = 1 \right\}. \tag{12}$$

In particular

$$|\mathscr{O}_n| = n^2 \prod_{p|n} \left(1 - \frac{1}{p^2}\right) = \varphi(n)\psi(n).$$

Thus $\mathrm{SL}_2(\mathbb{Z})$ operates transitively on the set $\mathbb{T}^2(n)$ of torsion points of *order* n. These are the torsion points in $\mathbb{T}^2[n]$ vanishing by multiplication with n, but do not vanish by multiplication with any $m|n$. We have

$$\mathscr{O}_n := \mathrm{SL}_2(\mathbb{Z}) \cdot [1/n] = \mathbb{T}^2(n).$$

To apply formula 9 we need to know how many points of $\mathscr{O}_n$ intersect with the line $\partial^{top}\mathscr{C}$ and this is easy, in fact:

$$\mathscr{O}_n \cap \partial^{top}\mathscr{C} = \{[a/n] \in \mathbb{T}^2 : \ a, n \in \mathbb{Z}, \ \gcd(a,n) = 1\}$$

and thus $|\mathscr{O}_n \cap \partial^{top}\mathscr{C}| = \varphi(n)$. Because of that $|\mathscr{O}_n \cap \mathscr{C}| = \psi(n)\varphi(n) - \varphi(n) = \varphi(n)(\psi(n) - 1)$. Now all horizontal cylinders on all the marked tori parameterized by $\mathscr{C}$ have width 1 and there are always two of them, while differentials on $\partial^{top}\mathscr{C}$ admit only one horizontal cylinder (of width one of course).

Altogether we find the asymptotic growth rate of periodic cylinders for any marked torus contained in $\mathscr{O}_n$:

$$c_{cyl}(n) = 2\frac{\varphi(n)(\psi(n) - 1)}{\psi(n)\varphi(n)} + \frac{\varphi(n)}{\psi(n)\varphi(n)} = 2 - \frac{1}{\psi(n)}. \tag{13}$$

Taking the limit for n to infinity gives the generic constant

$$c_{cyl}(gen) = \lim_{n\to\infty} c_{cyl}(n) = 2.$$

Counting saddle connections. The only interesting question about the quadratic growth rate for saddle connections connecting the two different marked

points. For each $0 < k < n$ with $(k, n) = 1$ we need to calculate the two distances of the point $[k/n] \in \mathbb{T}^2$ to $[0] \in \mathbb{T}^2$. Now using formula 10 we obtain for torsion points of order n:

$$c_{\pm}(n) = 2\frac{n^2}{\varphi(n)\psi(n)} \sum_{(k,n)=1} \frac{1}{k^2}. \tag{14}$$

In [S1] we gave an explicit argument showing that for differentials parameterized by irrational (= generic) points on $\mathbb{T}^2$:

$$c_{sc}(\pm) = \lim_{n\to\infty} c_{\pm}(n) = 2\zeta(2). \tag{15}$$

This example is the first of the series which we call *d-symmetric torus coverings.* To construct d-symmetric torus coverings one uses a connected sum construction for translation surfaces:

Connected sum construction. Given an Abelian differential (X, ω) and a leaf $\mathscr{L} \in \mathscr{F}_\theta(X)$. Take $a \in \mathscr{L}$ and define the line segment

$$I := [0, \epsilon]e^{i\theta} + a \subset \mathscr{L}.$$

Then for $d \geq 2$ and a cycle $\sigma \in S_d$ we define the Abelian differential

$$(\#^d_{I,\sigma}X, \#^d_{I,\sigma}\omega)$$

by slicing d named copies $X_1, ..., X_d$ of X along I and identify opposite sides of the slits according to the permutation σ. The differential $\#^d_{I,\sigma}\omega$ on $\#^d_{I,\sigma}X$ is uniquely defined by the property

$$\#^d_{I,\sigma}\omega|_{X_i} = \omega_i = \omega.$$

Note: we can rename the d copies of X such that the cycle σ becomes $\tau = (1, 2, 3, ..., d)$. In this case we simply write:

$$(\#^d_I X, \#^d_I \omega) = (\#^d_{I,\tau}X, \#^d_{I,\tau}\omega).$$

If γ_1 and γ_2 are two chains of geodesic segments on an Abelian differential (X, ω) with

$$\partial\gamma_1 = \partial\gamma_2$$

we might use cut and paste to see that

$$(\#^d_{\gamma_1}X, \#^d_{\gamma_1}\omega) = (\#^d_{\gamma_2}X, \#^d_{\gamma_2}\omega),$$

if γ_1 and γ_2 are isotopic along an isotopy fixing the endpoints $\partial\gamma_1$ and containing no cone points. Another way to say this is that the lifts of γ_i to the universal covering $\widetilde{X}$ of X bounds a disk B containing no cone points (in its interior).

d-symmetric differentials. We apply this construction to a torus covering, by taking d copies of $\mathbb{T}^2$, slice them along the projection of the line segment $I = I_v = [0, v] \subset \mathbb{C}$ ($v \in \mathbb{C}!$) to $\mathbb{T}^2$. Denote the resulting differential by

$$(\#^d_I\mathbb{T}^2, \#^d_I dz).$$

The underlying surface has genus d and the translation structure has precisely two cone points of order d. We define d-symmetric torus coverings as follows

- τ has exactly *two* zeros of order $d - 1$
- $\mathbb{Z}/d\mathbb{Z} \subset \mathrm{Aut}(Y, \tau)$
- $\deg(\pi) = \int_X \pi^*(dx \wedge dy) = d$

with the natural projection $\pi : (Y, \tau) \to \mathbb{C}/\operatorname{Per}(\tau)$. Note that all elliptic differentials of the shape $(\#_I^d \mathbb{T}^2, \#_I^d dz)$ are d-symmetric.

Denote the *set of isomorphy classes of d-symmetric coverings of* $\mathbb{T}^2 = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}i$ by $\mathscr{F}_d^{sym} := \mathscr{F}_d^{sym}(d-1, d-1)$. Note that our previous examples, 2-marked tori, are simply 1-symmetric differentials, and $\mathscr{F}_1(0,0) \cong \mathbb{T}^2 - \{0\}$. We show in [S3]:

Theorem 5. *The set $\mathscr{F}_d^{sym}$ has a natural structure as a torus (covering) $\mathbb{T}_d^2 := \mathbb{R}^2/d\mathbb{Z}^2$ without integer lattice points. The $\mathrm{SL}_2(\mathbb{Z})$ operation on $\mathscr{F}_d^{sym} = \mathbb{R}^2/d\mathbb{Z}^2 - \mathbb{Z}^2/d\mathbb{Z}^2$ commutes with the $\mathrm{SL}_2(\mathbb{Z})$ operation on surfaces parameterized by $\mathscr{F}_d^{sym}$.*

For d-symmetric differentials we evaluate formula 8 to find the Siegel-Veech constants:

Theorem 6. *Let $(S, \alpha) \in \mathbb{T}_d^2$ be d-symmetric and $(S, \alpha) \notin \mathbb{Q}^2/d\mathbb{Z}^2$, i.e. has infinite $\mathrm{SL}_2(\mathbb{Z})$-orbit in $\mathbb{T}_d^2$. Then the asymptotic quadratic growth rate of periodic cylinders c_{cyl} on S is:*

$$c_{cyl}(S) = c_{cyl}(d) = 2 \sum_{p|d} \frac{\varphi(p)}{p^3}. \tag{16}$$

Note that by Möbius inversion

$$\varphi(d) = \frac{d^3}{2} \sum_{p|d} \mu\left(\frac{d}{p}\right) c_{cyl}(p).$$

Using formula 9 we calculate the Siegel-Veech constants for d-symmetric differentials $(S, \alpha) \in \mathbb{Q}^2/d\mathbb{Z}^2 \subset \mathbb{T}_d^2$, the *torsion points* in $\mathscr{F}_d^{sym}$, as well. The various asymptotic constants depend very sensitive on the translation geometry of the surfaces. In particular some Siegel-Veech constants for special types of saddle connections are of interest.

3. Modular fibers in genus 2

How to describe modular fibers. Our approach works, if one is able to gain enough information about the translation geometry and topology of the space $\mathscr{F}_{\omega,\tau}$, in particular one needs to count certain sets of cone points of the space $\mathscr{F}_{\omega,\tau}$. We do not claim this is a trivial task, but one can do it to an extend making results access-able which are very hard to gain without using the geometry of $\mathscr{F}_{\omega,\tau}$.

For example: it takes a computer (program developed by G. Schmidthuesen [GS]) several days to calculate the index of the stabilizer of some differentials $(S, \alpha) \in \mathscr{F}_3(1,1)$ with small(!) $\mathrm{SL}_2(\mathbb{Z})$ orbit. On the other hand it takes not to long to make a picture of $\mathscr{F}_3(1,1)$, using cylinders contained in $\mathscr{F}_3(1,1)$. In case of finite orbit surfaces in $\mathscr{F}_3(1,1)$ it is possible to develop a formula for the order of the orbit [S4].

Absolute periods of $\mathscr{F}_3(1,1)$. The absolute period lattice $\operatorname{Per}(\omega_3)$ generated by the cylinders of $\mathscr{F}_3(1,1)$ is

$$\operatorname{Per}(\omega_3) = 2\mathbb{Z}^2 \subset \mathbb{Z}^2.$$

Different colors in Figure 1 show one possible tiling of $\mathscr{F}_3(1,1)$ by squares of size 2. We claim that $\operatorname{Per}(\omega_d) = 2\mathbb{Z}^2$ for all $d \geq 2$. Here is an indirect argument: In

[EMS] we found that $\mathscr{F}_d(1,1)$ is tiled by $\frac{1}{3}(d-1)d\varphi(d)\psi(d)$ unit squares. Taking the quotient with respect to the involution σ gives a map

$$\delta_d : \mathscr{F}_d(1,1)/\sigma \to \mathbb{CP}^1 = \mathbb{T}^2/(-\operatorname{id})$$

of degree

$$\deg(\delta_d) = \frac{1}{3}(d-1)d\varphi(d)\psi(d).$$

branched over the image of $0 = \mathbb{T}^2[1]$ under $\mathbb{T}^2 \to \mathbb{CP}^1$. Kani [Ka3] on the other hand describes a map

$$\hat{\delta}_d : \mathscr{F}_d(1,1)/\sigma \to \mathbb{CP}^1$$

of degree

$$\deg(\hat{\delta}_d) = \frac{1}{12}(d-1)d\varphi(d)\psi(d) = \frac{1}{4}\deg(\delta_d)$$

which is branched over the images of the 2-torsion points $\mathbb{T}^2[2]$ under $\mathbb{T}^2 \to \mathbb{CP}^1$. Here is a picture of the translation surface $\mathscr{F}_3(1,1)$ with some degenerated surfaces (vertices of the tiles of $\mathscr{F}_3(1,1)$) shown below.

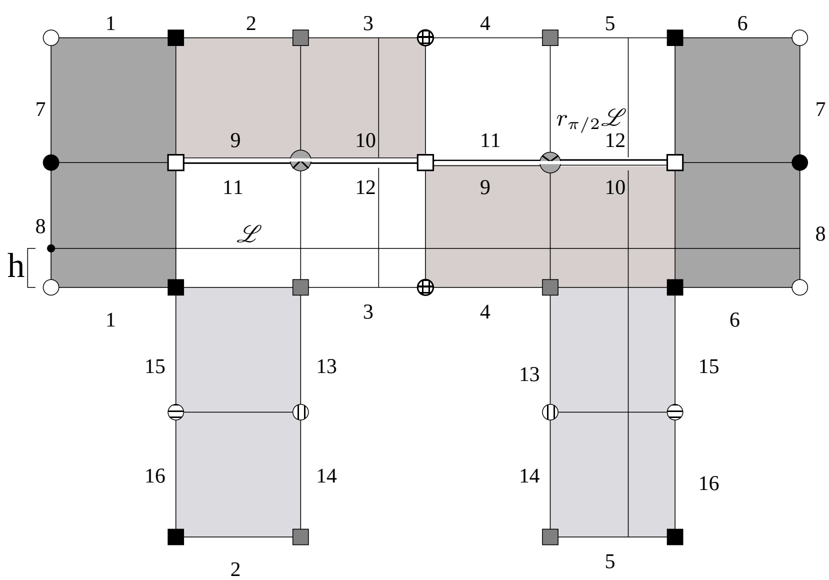

FIGURE 1. The modular surface $\mathscr{F}_3(1,1)$

Figure 2 presents the surfaces on the 'integer lattice' of $\mathscr{F}_3(1,1)$. The monodromy or identification scheme of each surface in the picture is as follows:

- *horizontal:* same color means same closed cylinder
- *vertical:* opposite sides are identified, unless something else is indicated by dashes.

The two degenerated surfaces sitting in the middle of the slit in Figure 1 are isomorphic, but appear as different points if one takes the closure of $(\mathscr{F}_3(1,1),\omega_3)$ as elliptic differential.

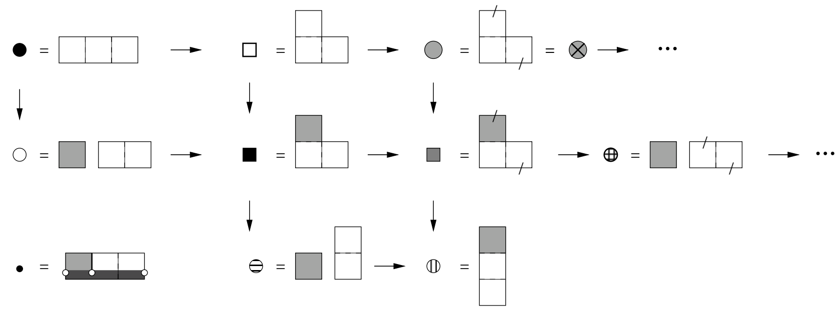

FIGURE 2. Surfaces on integer coordinates of $\mathscr{F}_3(1,1)$

Deforming along a loop $\mathscr{L}$ in $\mathscr{F}_3(1,1)$. Walking along the loop $\mathscr{L}$ in $\mathscr{F}_3(1,1)$ from the left to the right represents a deformation of the degree 3 torus cover denoted by the black dot to the right of the figure. The picture shows the 6 surfaces at the intersection points of $\mathscr{L}$ with the vertical edges of the tiling by squares.

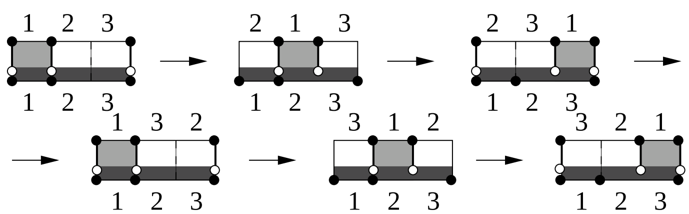

FIGURE 3. Deformation along $\mathscr{L}$

Note, while deforming a surface into its neighbor, the vertical gluing pattern changes by a transposition. Note also that $\mathscr{L}$ intersects $r_{\pi/2}\mathscr{L}$, its image under rotation by 90 degrees.

Now we present a picture of the quadratic differential q_3 ($\omega_3^2 = \mathrm{pr}_\sigma^* \, q_3$) on the sphere $\mathscr{F}_3(1,1)/\sigma$.

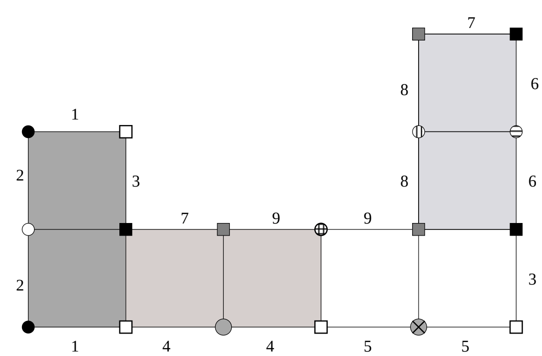

FIGURE 4. The flat sphere $\mathscr{F}_3(1,1)/\sigma$

Properties of $\mathscr{F}_\mathbf{d}(\mathbf{1},\mathbf{1})$. First we describe some translation surfaces belonging to $\mathscr{F}_d(1,1)$. With $\mathbb{T}^2(a,b) := \mathbb{C}/a\mathbb{Z} \oplus ib\mathbb{Z}$ and a line segment $I = I_v := [0,v] \subset \mathbb{C}$, $v \in \mathbb{C}$, we define the connected sum

$$\mathscr{S}_{a,v} := \mathbb{T}^2(a,1)\#_I\mathbb{T}^2(d-a,1) \in \mathscr{F}_d(1,1), \text{ where } (a,d) = 1.$$

The condition $(a,d) = \gcd(a,d) = 1$ is necessary and sufficient to make sure that $\mathscr{S}_{a,v}$ belongs to $\mathscr{F}_d(1,1)$, and not to a modular fiber of lower degree d. Now assume $v = t_h + it_v$ and $t_v \in (0,1)$. We call t_h the horizontal twist and t_v the vertical twist. Then the horizontal cylinder decomposition of $\mathscr{S}_{a,v}$ contains a cylinder of core width d above the first cone point, say z_0, and two cylinders of width a and $b = d-a$ on top of the second cone point $z_1 = [v]$. To the three horizontal cylinders we associate *twists*, given by $t_h(d) := t_h \mod d$ for the wide cylinder and by $t_h(a) := t_h \mod a$, $t_h(b) := t_h \mod b$ respectively, for the narrow cylinders. If we pick an integer twist t_h, the three twists essentially agree with the twist part of the coordinates for $\mathscr{F}_d(1,1)$, described in [EMS].

Loops and cylinders in $\mathscr{F}_\mathbf{d}(\mathbf{1},\mathbf{1})$. By condition $(a,d) = 1$ the Chinese remainder theorem implies the map

$$\begin{array}{ccc} t_h & \longmapsto & (t_h(a), t_h(b), t_h(c)) \\ \mathbb{R} & \to & \mathbb{R}/a\mathbb{Z} \oplus \mathbb{R}/b\mathbb{Z} \oplus \mathbb{R}/d\mathbb{Z} \end{array} \tag{17}$$

has kernel $abd\mathbb{Z} = a(d-a)d\mathbb{Z}$. From this it is easy to see the following

Proposition 1. *For all $t_v \in (0,1)$ and $0 < a < d$ with $(a,d) = 1$, the map*

$$\gamma : \mathbb{R}/abd\mathbb{Z} \ni t_h \mapsto \mathscr{S}_{a,t_h+it_v} \in \mathscr{F}_d(1,1)$$

is an isometrically embedded loop contained in the horizontal foliation of $\mathscr{F}_d(1,1)$. Moreover the image of

$$\gamma \times \mathrm{id} : \mathbb{R}/abd\mathbb{Z} \times (0,1) \ni (t_h, t_v) \mapsto \mathscr{S}_{a,t_h+it_v} \in \mathscr{F}_d(1,1)$$

is a maximal, horizontal cylinder $\mathscr{C}_a^+ \subset \mathscr{F}_d(1,1)$.

Remarks. A proof of this proposition on a formal level requires to introduce period coordinates for $\mathscr{F}_d(1,1)$ which we want to avoid at this place. Period coordinates are used in [S2]. However it is easy to check that the loop γ closes with $t_h = a(d-a)d$, if $(a,d) = 1$. The cylinder $\mathscr{C}_a^+$ is maximal because it is bounded by degenerate surfaces like $\mathscr{S}_{a,0}$ and $\mathscr{S}_{a,i}$, i.e. $t_h + it_v = 0$ or $t_h + it_v = i$. The '+ attached to $\mathscr{C}_a^+$ is because of our convention that the cone points z_o and z_1 are named. One obtains the cylinder $\mathscr{C}_a^- \subset \mathscr{F}_d(1,1)$ by taking

$$\mathscr{C}_a^- := \{\mathscr{S}_{a,t_h-it_v} : \ (t_h,t_v) \in \mathbb{R}/abd\mathbb{Z} \times (0,1)\}.$$

$\mathscr{U}$ action on $\mathscr{C}_1^+$. To establish connectedness of $\mathscr{F}_d(1,1)$ we look at the $\mathrm{SL}_2(\mathbb{Z})$ action on $\mathscr{F}_d(1,1)$. In particular we are interested in the action of

$$\mathscr{U} := \{u_n = \left[\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right] : n \in \mathbb{Z}\} \subset \mathrm{SL}_2(\mathbb{Z})$$

on $\mathscr{C}_1^\pm$. This looks trivial, but there is a nontrivial translation part caused by the $\mathscr{U}$ action on surfaces $\mathscr{S}_{1,t_h+it_v} \in \mathscr{C}_1^\pm$. In fact we have

$$u_1 \cdot \left[\begin{smallmatrix} t_h \\ t_v \end{smallmatrix}\right] = \left[\begin{smallmatrix} t_h+t_v-d \\ t_v \end{smallmatrix}\right].$$

or $u_1 \cdot \mathscr{S}_{1,t_h+it_v} = \mathscr{S}_{1,t_h+t_v-d+it_v}$. One can see this taking $t_v = 1$, i.e. $\mathscr{S}_{1,t_h+i}$, and using continuity of the $\mathrm{SL}_2(\mathbb{Z})$ action on $\mathscr{F}_d(1,1)$. This tells us in particular that

$\mathscr{U}$ really *acts on* $\mathscr{C}_1^+$.

Now we look to the action of the counter-clockwise rotation by $\pi/2$, i.e. $r_{\pi/2} := \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$. The identity

$$(r_{\pi/2} \cdot u_{d-1}) \cdot \mathscr{S}_{1,0} = r_{\pi/2} \cdot \mathscr{S}_{1,d-1} = \mathscr{S}_{1,1-d} \in \partial \mathscr{C}_1^+. \tag{18}$$

shows that $r_{\pi/2} \cdot \mathscr{C}_1^+$ intersects with $\mathscr{C}_1^+$ in an open set, since $\mathscr{S}_{1,1}$ is a smooth point.

Before we prove Theorem 2, we add information on the global structure of the space of all torus-coverings or *elliptic covers* $\mathscr{E}_d(1,1)$ of degree d, with two zeros of order one and absolute period lattice $\mathrm{Per}(\omega) = \Lambda$ of covolume 1. We have the following 'fiber-bundle' structure:

$$\mathscr{F}_d(1,1) \longrightarrow \mathscr{E}_d(1,1) \longrightarrow \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z}). \tag{19}$$

The base $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ parameterizes lattices Λ of covolume 1 and $\mathscr{F}_d(1,1)$ is the fiber over $\mathbb{Z} \oplus \mathbb{Z}i$. The other fibers are $\mathrm{SL}_2(\mathbb{R})$-deformations of $\mathscr{F}_d(1,1)$. We also need the following result

Theorem 7. [EMS] *The space $\mathscr{E}_d(1,1)$ is connected for all $d \geq 2$. In addition from each point in $\mathscr{F}_d(1,1) \subset \mathscr{E}_d(1,1)$ there is a path to the surface $\mathscr{S}_{a,i\epsilon}$ for an $0 < a < d$ with $(a,d) = 1$. In particular $\mathscr{F}_d(1,1)$ admits at most $\varphi(n)/2$ connected components.*

Now we can prove Theorem 2:

Proof. Assume $\mathscr{F}_d(1,1)$ is not connected. Since $\mathscr{E}_d(1,1)$ is connected by Theorem 7, all components of $\mathscr{F}_d(1,1)$ must be on a single $\mathrm{SL}_2(\mathbb{Z})$ orbit. In particular there is an affine map of $\mathscr{F}_d(1,1)$, induced by the $\mathrm{SL}_2(\mathbb{Z})$ action, which permutes the components of $\mathscr{F}_d(1,1)$. Now $\mathrm{SL}_2(\mathbb{Z})$ is generated by u_1 and $r_{\pi/2}$ and u_1 fixes $\mathscr{C}_1^+$, therefore it stabilizes a component of $\mathscr{F}_d(1,1)$. Because $r_{\pi/2}\mathscr{C}_1^+ \cap \mathscr{C}_1^+ \neq \emptyset$, $r_{\pi/2}$ stabilizes the same connected component and the statement follows. □

Remark 2. The above is a relatively simple strategy to show connectedness of fibers $\mathscr{F}$. To recall, take a loop $\mathscr{L}$ in the modular fiber and prove that it is stabilized by the parabolic map

$$u_v = \left\{ g \cdot \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \cdot g^{-1} : \; g \in \mathrm{SL}_2(\mathbb{Z}) \text{ with } \; g \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] = v \right\} \in \mathrm{SL}_2(\mathbb{Z})$$

fixing the holonomy-image $v = \mathrm{hol}(\mathscr{L}) \in \mathbb{R}^2$. Then show that the fiber containing $\mathscr{L}$ is stable under rotation by $r_{\pi/2}$. The method applies well in case the fiber of elliptic differentials is an orbit closure:

$$\mathscr{F}_\alpha := \overline{\mathrm{SL}_2(\mathbb{Z}) \cdot (S,\alpha)} \subset \overline{\mathrm{SL}_2(\mathbb{R}) \cdot (S,\alpha)} = \mathscr{E}_\alpha.$$

The argument is in general not sufficient, if the space of elliptic differentials is obtained by fixing algebraic or topological invariants of differentials. For example $\mathscr{E}_d(1,1)$ is given by taking all differentials (X,ω) of degree d, i.e. with canonical map $X \to \mathbb{C}/\mathrm{Per}(\omega)$ of degree d, and ω has precisely two zeros of order 1. In this case one needs to establish connectedness of the whole space $\mathscr{E}_d(1,1)$ first, see [EMS].

Finite $\mathbf{SL_2}(\mathbb{Z})$ orbits in $\mathscr{F}_\mathbf{3}(\mathbf{1},\mathbf{1})$. The next and final step is to classify all finite $\mathrm{SL}_2(\mathbb{Z})$-orbits contained in $\mathscr{F}_3(1,1)$. We will address this more generally in [S4]. After this is done the asymptotic formulæ can be evaluated if one is able to count

how many points on each orbit are contained in each horizontal cylinder of $\mathscr{F}_3(1,1)$. Since the generic constant for cylinders of periodic trajectories only depends on the horizontal cylinder decomposition of $\mathscr{F}_3(1,1)$, we can easily evaluate the generic asymptotic constant for $\mathscr{F}_3(1,1)$ and find:

$$c_{cyl}(gen) = \frac{1}{16}\left[12\left(\frac{1}{1}+\frac{1}{2^2}+\frac{1}{3^2}\right)+4\left(\frac{2}{1}+\frac{1}{2^2}\right)\right] = \frac{19}{12}. \tag{20}$$

In large d, the counting of horizontal cylinders in $\mathscr{F}_d(1,1)$ is non-trivial, see [EMS] for a coordinate approach.

Proof of Corollary 1. Suppose (X,ω) is an Abelian differential with (named) zeros z_i of order o_i, then the *Gauss-Bonnet formula* for translation surfaces says

$$\chi(X) = 2-2g(X) = -\sum_{i=1}^{n} o_i. \tag{21}$$

For quadratic differentials (Y,q) (with simple poles) there is a similar formula

$$2\chi(Y) = 4-4g(Y) = n_{-1} - \sum_{i=1}^{n} o_i, \tag{22}$$

where n_{-1} is the number of poles of q of order 1 and o_i is the order of the i-th zero z_i of q. A zero (pole) of order o_i is a cone point of total angle $(o_i+2)\pi$ w.r.t. the *half-translation* structure on (Y,q).

The expression for $\chi(\mathscr{F}_d^c(1,1))$ $(d \geq 3)$ comes from the fact that $\mathscr{F}_d^c(1,1)$ has $\frac{3}{8}(d-2)\varphi(d)\psi(d)$ cone-points, all of order 3. These cone-points are order 2 zeros of ω_d.

To calculate $\chi(\mathscr{F}_d^c(1,1)/\sigma)$ we note that q_d has $n_{+1} = \frac{3}{8}(d-2)\varphi(d)\psi(d)$ cone-points with total angle 3π, these are simple zeros of q_d. The number n_{-1} of simple poles of q_d equals the number of cone points of total angle π on $\mathscr{F}_d^c(1,1)/\sigma$, which in turn equals the number $N_{deg}(d)$ of *degenerated surfaces* in $\mathscr{F}_d^c(1,1)$. Thus we find the stated expression from

$$\begin{aligned} &2\chi(\mathscr{F}_d^c(1,1)/\sigma) = N_{deg}(d) - |Z(q_d)| = n_{-1} - n_{+1} = \\ &= \frac{1}{24}\left((5d+6)-9(d-2)\right)\varphi(d)\psi(d) = -\frac{1}{6}(d-6)\varphi(d)\psi(d) \quad \text{ for } \ d \geq 3. \end{aligned} \tag{23}$$

For $d=2$ we have $\chi(\mathscr{F}_2^c(1,1)/\sigma) = \chi(\mathbb{T}^2/(-\mathrm{id})) = \chi(\mathbb{CP}^1) = 0$.

Since q_d has only simple poles and zeros of order 1, Theorem 1.2 in [L] gives formula 7, after observing that

$$\frac{n_{-1}-n_{+1}}{4} = \frac{\chi(\mathscr{F}_d^c(1,1)/\sigma)}{2} = -\frac{1}{24}(d-6)\varphi(d)\psi(d) \in \mathbb{Z} \quad \text{ for } d \geq 3$$

and obvious simplifications when taking this expression modulo 2. □

Acknowledgments. The author is grateful for the atmosphere during the Activity on Algebraic and Topological Dynamics in Bonn. In particular I have to thank Sergij Kolyada for the excellent organization and Don Zagier for an additional short term invitation. While part of the activity I enjoyed discussions about the subject of this note with Anton Zorich, Curt McMullen, Richard Schwartz and Pascal Hubert, just to name a few. The author thanks the referee for many useful suggestions and for a careful reading of this partly overview article. Since some of the cited results, particularly the results about d-symmetric differentials, were not

peer-reviewed while this paper was under review, all mistakes are entirely the fault of the author.

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